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OPTIMIZATION, SYSTEM ANALYSIS, AND OPERATIONS RESEARCH

Pareto Set Design When Combining Feasible Solutions of the Multicriteria Axial Assignment Problem

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Abstract—This paper considers a two-criteria three-index axial assignment problem, representing a classical NP-hard problem even in the single-criterion case. Within this formulation, the problem of combining feasible solutions is posed; it is an assignment problem on the set of solutions containing only the components of the feasible solutions selected. A polynomial algorithm is proposed to find Pareto optimal solutions in the combination problem of two feasible solutions. Based on this algorithm, a heuristic approach is constructed to estimate the Pareto set of the multicriteria axial assignment problem.

Keywords: axial assignment problem, multi-index problems, multicriteria problems, Pareto optimality, combining solutions, polynomial solvability, NP-hardness

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1. INTRODUCTION

There is a wide class of applied problems formalized by multi-index axial assignment problems. Examples include resource allocation, scheduling, object tracking, robotic logistics [1–5], and others. The class of multi-index axial assignment problems is NP-hard even in the three-index case [6]. Particular polynomially solvable subclasses [1, 7] and subclasses with polynomial approximation algorithms [1, 8, 9] are known. The three-index axial assignment problem was studied in [10–16], including the construction of approximate approaches and the implementation of exact algorithms. Multi-criteria formulations of multi-index assignment problems were discussed in [17–20].

This paper addresses the issue of constructing the Pareto set in a two-criteria three-index axial assignment problem. We formulate the problem of combining feasible solutions, which is a multicriteria assignment problem on the set of solutions obtained by combining the components of given feasible solutions. We propose a polynomial algorithm for constructing a subset of Pareto optimal solutions on the set of combinations of two feasible solutions. This algorithm is applied to develop a heuristic algorithm for approximating the Pareto set of the two-criteria three-index assignment problem. A computational experiment is carried out to illustrate the approach. The paper continues the series of research works [21–24] devoted to combining the solutions of the axial assignment problem.

The remainder of this paper is organized as follows. In Section 2, we formulate the two-criteria three-index axial assignment problem and the corresponding combination problem of feasible solutions. Section 3 describes an algorithm for finding a subset of Pareto optimal solutions of the combination problem. The results of computational experiments are provided in Section 4.

2. PROBLEM STATEMENT

Let I, J, and K be disjoint index sets, $I \cap J = \emptyset$, $I \cap K = \emptyset$, $J \cap K = \emptyset$ and $|I| = |J| = |K| = n$; c_{ijk} and d_{ijk}, where $i \in I$, $j \in J$, and $k \in K$, are three-index cost matrices; finally, x_{ijk}, where $i \in I$, $j \in J$, and $k \in K$, is the three-index matrix of the variables. Then the two-criteria three-index axial assignment problem is formulated as the following integer linear programming problem:

$$\sum_{i\in I}\sum_{j\in J} x_{ijk} = 1, \quad k \in K, \tag{1}$$

$$\sum_{i\in I}\sum_{k\in K} x_{ijk} = 1, \quad j \in J, \tag{2}$$

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} = 1, \quad i \in I, \tag{3}$$

$$x_{ijk} \in \{0,1\}, \quad i \in I, \quad j \in J, \quad k \in K, \tag{4}$$

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ijk}x_{ijk} \to \min, \tag{5}$$

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} d_{ijk}x_{ijk} \to \min. \tag{6}$$

For the sake of convenience, we denote by Z_2 the two-criteria problem (1)–(6) and introduce

$$C(x) = \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ijk}x_{ijk},$$

$$D(x) = \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} d_{ijk}x_{ijk}.$$

Then x^* is called a Pareto optimal solution of the problem Z_2 if x^* satisfies the constraints (1)–(4) and there is no x' satisfying (1)–(4) such that $C(x^*) > C(x')$ and $D(x^*) \geqslant D(x')$ or $C(x^*) \geqslant C(x')$ and $D(x^*) > D(x')$.

As is well known, the (single-criterion) axial assignment problem (1)–(5) is NP-hard [6]. Hence, the problem of constructing a Pareto optimal solution of the two-criteria axial assignment problem Z_2 is also NP-hard. To prove this fact, we need to define $c_{ijk} = d_{ijk}$, $i \in I$, $j \in J$, $k \in K$, in the corresponding problem Z_2.

Proposition 1. *The problem of constructing a Pareto optimal solution of the problem Z_2 is NP-hard.*

Thus, it is topical to develop effective heuristic approaches to estimating the Pareto set of the problem Z_2. The heuristic approach proposed below is based on combining feasible solutions of the problem Z_2. Here, combining means solving the problem on the set of solutions containing only the components of given feasible solutions.

Now we state the combination problem. Let a given set $W \subset I \times J \times K$ define a subset of allowed assignments. We introduce the auxiliary constraint

$$x_{ijk} = 0, \quad (i,j,k) \notin W. \tag{7}$$

The two-criteria problem (1)–(4), (7), (5), (6) for a given set W will be denoted by $Z_2(W)$. Obviously, problem (1)–(6) corresponds to the problem $Z_2(I \times J \times K)$.

Then x^* is called a Pareto optimal solution of the problem $Z_2(W)$ if x^* satisfies the constraints (1)–(4), (7) and there is no x' satisfying (1)–(4), (7) such that $C(x^*) > C(x')$ and $D(x^*) \geqslant D(x')$ or $C(x^*) \geqslant C(x')$ and $D(x^*) > D(x')$.

Checking the consistency of all constraints in the problem $Z_2(W)$ with an arbitrary set W, i.e., the system (1)–(4), (7), is an NP-complete problem [1]. When solving the combination problem, we will consider the sets W corresponding to the assignments of a given subset of feasible solutions.

Let x_{ijk}, where $i \in I$, $j \in J$, and $k \in K$, be a feasible solution of the system of constraints (1)–(4). Then $W(x)$ will denote the following set of allowed assignments:

$$W(x) = \{(i,j,k) | x_{ijk} = 1, i \in I, j \in J, k \in K\}.$$

Consider $x^1_{ijk}, x^2_{ijk}, \ldots, x^r_{ijk}$, where $i \in I$, $j \in J$, and $k \in K$, representing arbitrary r feasible solutions of the system of constraints (1)–(4). Then

$$W(x^1, x^2, \ldots, x^r) = W(x^1) \cup W(x^2) \cup \ldots \cup W(x^r).$$

Below, we will investigate the problem $Z_2(W(x^1, x^2, \ldots, x^r))$.

3. CONSTRUCTING A SUBSET OF PARETO OPTIMAL SOLUTIONS

For $r = 2$, let x^1 and x^2 be given feasible solutions of the system of constraints (1)–(4). Consider the problem $Z_2(W(x^1, x^2))$ and the issue of constructing its Pareto optimal solutions.

Algorithm 1. Constructing a subset of Pareto optimal solutions of the problem $Z_2(W(x^1, x^2))$.

Step 1. Construct a graph $G = (V, A)$, where

$$V = \{I \cup J \cup K\}, \quad A = \Big\{(i,j), (i,k), (j,k) | (i,j,k) \in W(x^1, x^2)\Big\}.$$

Step 2. Find the connected components V_l, $l = \overline{1,q}$, of the graph G and construct the subgraphs $G_l = (V_l, A_l)$, $l = \overline{1,q}$, generated by the corresponding connected components.

Step 3. Construct the sets

$$D^1_l = \Big\{(i,j,k) | (i,j,k) \in W(x^1), (i,j), (i,k), (j,k) \in A_l\Big\}, \quad l = \overline{1,q},$$

$$D^2_l = \Big\{(i,j,k) | (i,j,k) \in W(x^2), (i,j), (i,k), (j,k) \in A_l\Big\}, \quad l = \overline{1,q}.$$

Step 4. Let

$$P_l = \left\{ p \,\middle|\, \sum_{(i,j,k) \in D^p_l} c_{ijk} = \min_{p' \in \{1,2\}} \sum_{(i,j,k) \in D^{p'}_l} c_{ijk},\ p \in \{1,2\} \right\},$$

$$p_l = \operatorname*{argmin}_{p \in P_l} \sum_{(i,j,k) \in D^p_l} d_{ijk}, \quad l = \overline{1,q}.$$

Step 5. Construct the Pareto optimal solution x^{*0} using the following algorithm. Let $x^{*0}_{ijk} := 0$, where $i \in I$, $j \in J$, and $k \in K$. For each $l = \overline{1,q}$, execute $x^{*0}_{ijk} := 1, (i,j,k) \in D^{p_l}_l$.

Step 6. Construct the following index set of the connected components:

$$L = \left\{ l \middle| \sum_{(i,j,k)\in D_l^{\overline{p_l}}} c_{ijk} > \sum_{(i,j,k)\in D_l^{p_l}} c_{ijk} \quad \text{and} \quad \sum_{(i,j,k)\in D_l^{\overline{p_l}}} d_{ijk} < \sum_{(i,j,k)\in D_l^{p_l}} d_{ijk},\ l \in \{1,\dots,q\} \right\},$$

where $\overline{p} = 3 - p$ for $p \in \{1, 2\}$.

Step 7. Sort this set in ascending order of the value

$$tg(l) = \frac{\sum\limits_{(i,j,k)\in D_l^{p_l}} d_{ijk} - \sum\limits_{(i,j,k)\in D_l^{\overline{p_l}}} d_{ijk}}{\sum\limits_{(i,j,k)\in D_l^{\overline{p_l}}} c_{ijk} - \sum\limits_{(i,j,k)\in D_l^{p_l}} c_{ijk}},$$

i.e., let $L = \{l_1, \dots, l_{|L|}\}$ and $tg(l_s) \geqslant tg(l_{s+1}),\ s = \overline{1, |L| - 1}$.

Step 8. Construct $|L|$ Pareto optimal solutions x^{*s}, $s = \overline{1, |L|}$, in the following way. Let $x^{*s}_{ijk} := 0,\ i \in I,\ j \in J,\ k \in K$ and, for each $t = \overline{1, |q|}$, execute:

if $t \in \{l_1, \dots, l_s\}$, then $x^{*s}_{ijk} := 1, (i, j, k) \in D_t^{\overline{p_t}}$;

otherwise, $x^{*s}_{ijk} := 1, (i, j, k) \in D_t^{p_t}$.

Here is a numerical example for the operation of Algorithm 1.

Example 1. An illustration of Algorithm 1.

Let $n = 6$ and let the matrices c_{ijk} and d_{ijk} be

$$c_{ijk} = \begin{cases} 0, (i,j,k) \in \{(1,1,1),(1,1,2),(2,2,1),(3,3,3),(4,4,4),(3,3,4), \\ \qquad\qquad (5,5,5),(5,5,6),(6,6,5)\}, \\ 1, (i,j,k) \in \{(4,4,3)\}, \\ 2, (i,j,k) \in \{(6,6,6)\}, \\ 3, (i,j,k) \in \{(2,2,2)\}, \\ 10 \text{ otherwise.} \end{cases}$$

$$d_ijk = \begin{cases} 0, (i,j,k) \in \{(1,1,1),(1,1,2),(2,2,2),(3,3,3),(4,4,3),(3,3,4), \\ \qquad\qquad (5,5,5),(5,5,6),(6,6,6)\}, \\ 1, (i,j,k) \in \{(2,2,1),(4,4,4)\}, \\ 4, (i,j,k) \in \{(6,6,5)\}, \\ 10 \text{ otherwise,} \end{cases}$$

Consider two feasible solutions x^1 and x^2 of the form

$$x^1_{ijk} = \begin{cases} 1, (i,j,k) \in \{(1,1,1),(2,2,2),(3,3,3),(4,4,4),(5,5,5),(6,6,6)\}, \\ 0 \text{ otherwise,} \end{cases}$$

$$x^2_{ijk} = \begin{cases} 1, (i,j,k) \in \{(1,1,2),(2,2,1),(3,3,4),(4,4,3),(5,5,6),(6,6,5)\}, \\ 0 \text{ otherwise.} \end{cases}$$

We describe the operation of Algorithm 1 on this example.

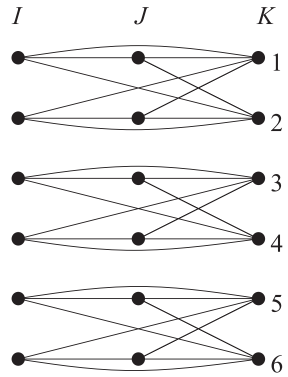

Fig. 1. The graph G at Step 1.

Figure 1 shows the graph G yielded by Step 1.

At Step 2 the connected components of the graph G are constructed. Obviously, the graph G has three connected components of the same structure. The sets obtained at Step 3 have the form

$$D_1^1 = \{(1,1,1),(2,2,2)\},\quad D_2^1 = \{(3,3,3),(4,4,4)\},\quad D_3^1 = \{(5,5,5),(6,6,6)\},$$
$$D_1^2 = \{(1,1,2),(2,2,1)\},\quad D_2^2 = \{(3,3,4),(4,4,3)\},\quad D_3^2 = \{(5,5,6),(6,6,5)\}.$$

At Step 4 we obtain

$$P_1 = \{2\},\quad p_1 = 2,$$
$$P_2 = \{1\},\quad p_2 = 1,$$
$$P_3 = \{2\},\quad p_3 = 2.$$

At Step 5 the first Pareto optimal solution x^{*0} is constructed as follows:

$$x_{ijk}^{*0} = \begin{cases} 1, (i,j,k) \in \{(1,1,2),(2,2,1),(3,3,3),(4,4,4),(5,5,6),(6,6,5)\}, \\ 0 \text{ otherwise.} \end{cases}$$

The criteria values achieved at this solution are $C(x^{*0}) = 0$ and $D(x^{*0}) = 6$.

At Step 6 the set L contains all three connected components; at Step 7 it will be sorted to $l_1 = 3$, $l_2 = 2$, and $l_3 = 1$. The next three Pareto optimal solutions constructed at Step 8 are

$$x_{ijk}^{*1} = \begin{cases} 1, (i,j,k) \in \{(1,1,2),(2,2,1),(3,3,3),(4,4,4),(5,5,5),(6,6,6)\}, \\ 0 \text{ otherwise,} \end{cases}$$

$$x_{ijk}^{*2} = \begin{cases} 1, (i,j,k) \in \{(1,1,2),(2,2,1),(3,3,4),(4,4,3),(5,5,5),(6,6,6)\}, \\ 0 \text{ otherwise,} \end{cases}$$

$$x_{ijk}^{*3} = \begin{cases} 1, (i,j,k) \in \{(1,1,1),(2,2,2),(3,3,4),(4,4,3),(5,5,5),(6,6,6)\}, \\ 0 \text{ otherwise.} \end{cases}$$

The corresponding criteria values are

$$C(x^{*1}) = 2,\quad D(x^{*1}) = 2,$$
$$C(x^{*2}) = 3,\quad D(x^{*2}) = 1,$$
$$C(x^{*3}) = 6,\quad D(x^{*3}) = 0.$$

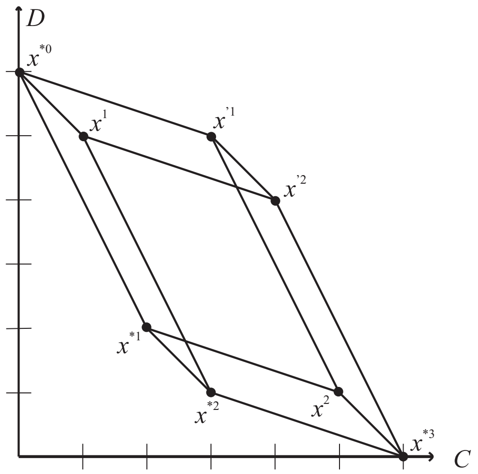

Fig. 2. The solutions found on the CD plane.

The resulting solutions are displayed on the CD plane, according to the value of their criterion (Fig. 2). Here, x^1 and x^2 are the initial solutions; x^{*0}, x^{*1}, x^{*2}, and x^{*3} are the Pareto optimal solutions of the combination problem obtained using Algorithm 1; finally, x'^1 and x'^2 are feasible solutions of the combination problem. In this example, there is one more Pareto optimal solution x^1 in addition to those found by the algorithm.

Next, we prove the correctness of Algorithm 1.

Consider the multicriteria problem. Let $X \subseteq R^n$ be the set of feasible solutions of the problem and $\overrightarrow{f(x)} \in R^m$ be the objective function of the problem, $f_i(x) \to \min$, $i = \overline{1, m}$.

Proposition 2. *Let a set $X \subseteq R^n$ be finite. If $x \in X$ is not a Pareto optimal solution, then there exists a Pareto optimal solution $y \in X$ dominating x, i.e., $f_i(y) \leqslant f_i(x)$, $i = \overline{1, m}$, and there exists an index $j \in \{1, \ldots, m\}$ such that $f_j(y) < f_j(x)$.*

Proof. Under the hypotheses of this proposition, suppose that $x \in X$ is not a Pareto optimal solution. In this case, we construct an algorithm for obtaining such a solution y.

Step 1. Find a solution x' dominating the solution x. If no such solution existed, the solution x would be Pareto optimal (a contradiction).

Step 2. Let $x := x'$ if x is Pareto-optimal, and return x; otherwise, go to Step 1.

By the definition of Pareto optimality, the total number of Steps 2 of the algorithm will not exceed the cardinality of the solution set. Since the set is finite, the algorithm will output a Pareto optimal solution in a finite number of steps. The proof is complete.

Proposition 3. *Any feasible solution x satisfying the system of constraints of the problem $Z_2(W(x^1, x^2))$ can be constructed by choosing triples only from the first or only from the second solution for each connected component independently.*

Proof. This result was established when proving Theorem 1 in [21].

Therefore, we have

$$p_l(x) = \begin{cases} 1 \text{ if } x_{ijk} = 1 \text{ for all } (i,j,k) \in D_l^1 \\ 2 \text{ if } x_{ijk} = 1 \text{ for all } (i,j,k) \in D_l^2. \end{cases}$$

Proposition 4. *For any Pareto optimal solution x' of the problem $Z_2(W(x^1, x^2))$,*

$$\sum_{(i,j,k)\in D_l^{p_l(x')}} c_{ijk} = \sum_{(i,j,k)\in D_l^{p_l(x^{*0})}} c_{ijk} \quad \textit{and} \quad \sum_{(i,j,k)\in D_l^{p_l(x')}} d_{ijk} = \sum_{(i,j,k)\in D_l^{p_l(x^{*0})}} d_{ijk}, \quad l \notin L.$$

Proof. Assume on the contrary the existence of a Pareto optimal solution x' with an index $l \notin L$ such that

$$\sum_{(i,j,k)\in D_l^{p_l(x')}} c_{ijk} \neq \sum_{(i,j,k)\in D_l^{p_l(x^{*0})}} c_{ijk} \quad \text{or} \quad \sum_{(i,j,k)\in D_l^{p_l(x')}} d_{ijk} \neq \sum_{(i,j,k)\in D_l^{p_l(x^{*0})}} d_{ijk}.$$

Consider two cases.

1. $\sum\limits_{(i,j,k)\in D_l^{p_l(x')}} c_{ijk} \neq \sum\limits_{(i,j,k)\in D_l^{p_l(x^{*0})}} c_{ijk}.$

According to Step 5 of Algorithm 1,

$$\sum_{(i,j,k)\in D_l^{p_l(x')}} c_{ijk} > \sum_{(i,j,k)\in D_l^{p_l(x^{*0})}} c_{ijk}.$$

Then:

a) If $\sum\limits_{(i,j,k)\in D_l^{p_l(x')}} d_{ijk} < \sum\limits_{(i,j,k)\in D_l^{p_l(x^{*0})}} d_{ijk}$, we have $l \in L$ (a contradiction).

b) Otherwise, $\sum\limits_{(i,j,k)\in D_l^{p_l(x')}} d_{ijk} \geqslant \sum\limits_{(i,j,k)\in D_l^{p_l(x^{*0})}} d_{ijk}$, and the triples corresponding to connected component l in the solution x' can be replaced as follows:

$$x'_{ijk} := 0,\ (i,j,k) \in D_l^{p_l(x')},$$
$$x'_{ijk} := 1,\ (i,j,k) \in D_l^{p_l(x^{*0})}.$$

The values of both criteria will be reduced accordingly; therefore, x' is not a Pareto optimal solution (a contradiction).

2. $\sum\limits_{(i,j,k)\in D_l^{p_l(x')}} c_{ijk} = \sum\limits_{(i,j,k)\in D_l^{p_l(x^{*0})}} c_{ijk}$, $\sum\limits_{(i,j,k)\in D_l^{p_l(x')}} d_{ijk} \neq \sum\limits_{(i,j,k)\in D_l^{p_l(x^{*0})}} d_{ijk}.$

According to Step 5 of Algorithm 1,

$$\sum_{(i,j,k)\in D_l^{p_l(x')}} d_{ijk} > \sum_{(i,j,k)\in D_l^{p_l(x^{*0})}} d_{ijk},$$

and the triples corresponding to connected component l in the solution x' can be replaced as follows:

$$x'_{ijk} := 0,\ (i,j,k) \in D_l^{p_l(x')},$$
$$x'_{ijk} := 1,\ (i,j,k) \in D_l^{p_l(x^{*0})}.$$

Again, the values of both criteria will be reduced, and hence x' is not a Pareto optimal solution (a contradiction). The proof of this proposition is complete.

We denote by P_t, $t = \overline{1,|L|}$, the straight line connecting the points x^{*t} and x^{*t+1} on the CD plane. The equation of P_t has the form

$$\frac{D - D(x^{*t})}{D(x^{*t-1}) - D(x^{*t})} = \frac{C - C(x^{*t})}{C(x^{*t-1}) - C(x^{*t})}.$$

For the sake of convenience, it can be reduced to

$$D = D(x^{*t}) + \frac{D(x^{*t}) - D(x^{*t-1})}{C(x^{*t-1}) - C(x^{*t})}(C(x^{*t}) - C) = D(x^{*t}) + tg(l_t)(C(x^{*t}) - C).$$

Also, this equation is equivalently written as

$$D = D(x^{*t-1}) + tg(l_t)(C(x^{*t-1}) - C).$$

We say that a feasible solution x of the problem $Z_2(W(x^1, x^2))$ is not below the straight line P_t if

$$D(x) \geqslant D(x^{*t}) + tg(l_t)(C(x^{*t}) - C(x)).$$

Obviously, if a solution x is not below the straight line P_t, it will not dominate any of the solutions x^{*t-1}, x^{*t}.

Proposition 5. *If for some $e \in \{1, \ldots, |L|\}$ a solution x is not below the straight line P_e and $C(x) \leqslant C(x^{*e})$, then it is not below any straight line $P_t,\ t = \overline{e, |L|}$.*

Proof. Since x is not below the straight line P_e, we have

$$D(x) \geqslant D(x^{*e}) + tg(l_e)(C(x^{*e}) - C(x)).$$

By construction, $tg(l_t) \geqslant tg(l_{t+1}) \geqslant 0,\ t = \overline{1, |L| - 1}$. Due to the hypothesis of this proposition, $(C(x^{*e}) - C(x)) \geqslant 0$. Therefore,

$$tg(l_e)(C(x^{*e}) - C(x)) \geqslant tg(l_{e+1})(C(x^{*e}) - C(x))$$

and consequently,

$$D(x) \geqslant D(x^{*e}) + tg(l_{e+1})(C(x^{*e}) - C(x)).$$

In other words, the solution x is not below the straight line P_{e+1}. By induction, we establish that x is not below the straight line P_t, $t \geqslant e$. The proof is complete.

Theorem 1. *There exists no Pareto optimal solution x' of the problem $Z_2(W(x^1, x^2))$ that dominates any of the solutions constructed by Algorithm 1.*

Proof. Assume on the contrary the existence of a Pareto optimal solution x' dominating at least one of the solutions $x^{*0}, \ldots, x^{*|L|}$ constructed by the algorithm. Due to Proposition 4, such a solution can differ from x^{*0} only in the triples of components $l \in L$.

We construct $|L| + 1$ solutions $x''^s,\ s = \overline{0, |L|}$, as follows.

Step 1. Let $x''^s_{ijk} := 0,\ i \in I,\ j \in J,\ k \in K$.

Step 2. For each $t = \overline{1, q}$, execute:

— If $t \in \{l_1, \ldots, l_s\}$ and $p_t(x') \neq p_t(x^{*0})$, then $x''^s_{ijk} := 1,\ (i, j, k) \in D_t^{\overline{p_t}}$.

— Otherwise, $x''^s_{ijk} := 1,\ (i, j, k) \in D_t^{p_t}$.

By construction, $x''^0 = x^{*0}$, and hence the solution x''^0 is not below the straight line P_1.

Now we demonstrate that if the solution x''^s is not below the straight lines $P_t,\ t = \overline{1, |L|}$, then the solution x''^{s+1} is not below the straight lines $P_t,\ t = \overline{1, |L|}$, as well.

For each straight line $P_t,\ t = \overline{1, r + 1}$, two cases are possible:

1. $p_{l_{s+1}}(x') = p_{l_{s+1}}(x^{*0})$, then the solution x''^{s+1} coincides with the solution x''^s; therefore, it is not below the straight line P_t.

2. $p_{l_{s+1}}(x') \neq p_{l_{s+1}}(x^{*0})$, then the condition that x''^s is not below P_t implies

$$D(x''^s) \geqslant D(x^{*t}) + tg(l_t)(C(x^{*t}) - C(x''^s)). \tag{8}$$

By construction,

$$D(x''^{s+1}) = D(x''^{s}) + tg(l_{s+1})(C(x''^{s}) - C(x''^{s+1})).$$

Applying inequality (8) yields

$$D(x''^{s+1}) \geqslant D(x^{*t}) + tg(l_t)(C(x^{*t}) - C(x''^{s})) + tg(l_{s+1})(C(x''^{s}) - C(x''^{s+1}))$$

and consequently,

$$D(x''^{s+1}) \geqslant D(x^{*t}) + tg(l_t)(C(x^{*t}) - C(x''^{s+1})) + (tg(l_t) - tg(l_{s+1}))(C(x''^{s+1}) - C(x''^{s})).$$

By construction $C(x''^{s+1}) - C(x''^{s}) \geqslant 0$ and $tg(l_t) - tg(l_{s+1}) \geqslant 0$ for $s+1 \geqslant t$. Then

$$D(x''^{s+1}) \geqslant D(x^{*t}) + tg(l_t)(C(x^{*t}) - C(x''^{s+1})),$$

i.e., the solution x''^{s+1} is not below the straight line P_t.

Hence, if the solution x''^{s} is not below the straight lines $P_t,\ t = \overline{1, |L|}$, then the solution x''^{s+1} is not below the straight lines $P_t,\ t = \overline{1, s+1}$.

By construction, $C(x''^{s+1}) \leqslant C(x^{*s+1})$. Then, according to Proposition 5, the solution x''^{s+1} is not below the straight lines P_t, $t = \overline{s+1, |L|}$ (here $e = s+1$). Therefore, if the solution x''^{s} is not below the straight lines P_t, $t = \overline{1, |L|}$, then the solution x''^{s+1} is not below the straight lines P_t, $t = \overline{1, |L|}$. By induction, we establish that $x''^{|L|}$ is not below the straight lines $P_t,\ t = \overline{1, |L|}$.

By construction, $x''^{|L|} = x'$, i.e., x' is not below the straight lines P_t, $t = \overline{1, |L|}$; hence, the solution x' dominates neither of the solutions $x^{*0}, \ldots, x^{*|L|}$ found by Algorithm 1. This contradiction completes the proof of Theorem 1.

Theorem 2. *The solutions $x^{*0}, \ldots, x^{*|L|}$ obtained using Algorithm 1 are Pareto optimal in the problem $Z_2(W(x^1, x^2))$.*

Proof. Assume on the contrary the existence of a solution $x^{*q'}$, $q' \in \{0, \ldots, L\}$, that is not Pareto optimal in the problem $Z_2(W(x^1, x^2))$. Then, according to Proposition 2, there is a Pareto optimal solution x' of the problem $Z_2(W(x^1, x^2))$ dominating $x^{*q'}$:

$$C(x') < C(x^{*q'}) \ \text{ and } \ D(x') < D(x^{*q'}).$$

However, due to Theorem 1, such a solution x' does not exist. This contradiction completes the proof of Theorem 2.

Proposition 6. *Algorithm 1 does not necessarily yield the entire set of Pareto optimal solutions of the problem $Z_2(W(x^1, x^2))$.*

Proof. We provide a corresponding numerical example. Let $n = 4$ and

$$x^1_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in \{(1,1,1), (2,2,2), (3,3,3), (4,4,4)\} \\ 0 \text{ otherwise,} \end{cases}$$

$$x^2_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in \{(1,1,2), (2,2,1), (3,3,4), (4,4,3)\} \\ 0 \text{ otherwise.} \end{cases}$$

In addition, let

$$c_{ijk} = \begin{cases} 0 \text{ if } (i,j,k) \in \{(1,1,1), (2,2,2), (3,3,3), (4,4,4), (2,2,1), (4,4,3)\} \\ 1 \text{ if } (i,j,k) = (1,1,2) \\ 2 \text{ if } (i,j,k) = (3,3,4) \\ 5 \text{ otherwise,} \end{cases}$$

$$d_{ijk} = \begin{cases} 0 \text{ if } (i,j,k) \in \{(2,2,2),(4,4,4),(1,1,2),(2,2,1),(3,3,4),(4,4,3)\} \\ 1 \text{ if } (i,j,k) = (1,1,1) \\ 2 \text{ if } (i,j,k) = (3,3,3) \\ 5 \text{ otherwise.} \end{cases}$$

Then there exist four feasible solutions of the combination problem:

$$x^{*1}_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in \{(1,1,1),(2,2,2),(3,3,3),(4,4,4)\} \\ 0 \text{ otherwise,} \end{cases}$$

where $C(x^{*1}) = 0$ and $D(x^{*1}) = 3$;

$$x^{*2}_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in \{(1,1,2),(2,2,1),(3,3,4),(4,4,3)\} \\ 0 \text{ otherwise,} \end{cases}$$

where $C(x^{*2}) = 3$ and $D(x^{*2}) = 0$;

$$x^{*3}_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in \{(1,1,2),(2,2,1),(3,3,3),(4,4,4)\} \\ 0 \text{ otherwise,} \end{cases}$$

where $C(x^{*3}) = 1$ and $D(x^{*3}) = 2$; and finally,

$$x^{*4}_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in \{(1,1,1),(2,2,2),(3,3,4),(4,4,3)\} \\ 0 \text{ otherwise,} \end{cases}$$

where $C(x^{*4}) = 2$ and $D(x^{*4}) = 1$.

Each of the four solutions constructed is Pareto optimal in the problem $Z_2(W(x^1, x^2))$. Note that $q = 2$ and $|L| = 2$; hence, Algorithm 1 will find only three solutions. The proof of this proposition is complete.

Thus, Algorithm 1 constructs Pareto optimal solutions of the problem $Z_2(W(x^1, x^2))$ without any guarantee of obtaining the entire Pareto set.

Theorem 3. *Algorithm 1 requires $O(n^2)$ computational operations.*

Proof. Let the input data of Algorithm 1 be feasible solutions x^1 and x^2 represented as a collection of triples (i,j,k) and costs c_{ijk} for which the corresponding variables are 1. According to (1)–(4), the number of such triples is n for each feasible solution. Step 1 of the algorithm is to construct a graph $G = (V, A)$, where $|V| = O(n)$ and $|A| = O(n)$. At Step 2 the graph G is partitioned into connected components. This partition is obtained using the width-first traversal of the graph, which requires $O(|V| + |A|) = O(n)$ computational operations. Step 3 serves to determine the corresponding connected components for the input triples of the algorithm; this step requires $O(n)$ computational operations. At Step 4, for each connected component, the algorithm determines from which solution the triples will be taken to the first Pareto optimal solution; this step requires $O(n)$ computational operations. Step 5 is to construct the first Pareto optimal solution; since the solution can be represented as a collection of n triples (i,j,k), this step requires $O(n)$ computational operations. At Step 6 only the connected components matching the criterion are selected from all such components, which requires $O(n)$ computational operations. Step 7 serves to sort the subset of the connected components, which requires $O(n\log(n))$ computational operations. Finally, at Step 8, $O(n)$ solutions are constructed, each in $O(n)$ computational operations. Thus, Algorithm 1 requires $O(n^2)$ computational operations. The proof of the theorem is complete.

4. COMPUTATIONAL EXPERIMENTS

Algorithm 1 yields a subset of the Pareto set in the problem $Z_2(W(x^1, x^2))$. Let us apply this algorithm to develop heuristic approximation methods for the Pareto set of the original problem Z_2. The effectiveness of the heuristic algorithm will be assessed via computational experiments.

Experiment 1. Construct the matrices c_{ijk} and d_{ijk} in the following way: for each index $i \in I \cup J \cup K$, generate a random point p on the XY plane so that p_x and p_y are integer and uniformly distributed on the closed interval $[0, 2^{32} - 1]$. Then $c_{ijk} = dist(i, j) + dist(j, k) + dist(i, k)$, where $dist(a, b)$ indicates the Manhattan distance between points a and b. Determine d_{ijk} by analogy.

A local optimization procedure for a feasible solution x of the problem Z_2 includes several steps as follows.

Step 1. Choose a random number $a \in [0, 1]$ equiprobably.

Step 2. Construct the three-index cost matrix $e_{ijk} = ac_{ijk} + (1 - a)d_{ijk}$.

Step 3. Apply the local optimization procedure proposed in [15] to the solution x of the original problem, but with the criterion $\sum_{i\in I} \sum_{j\in J} \sum_{k\in K} e_{ijk}x_{ijk} \to \min$.

We construct n random feasible solutions $x_1, \ldots, x_n$ of the problem Z_2, and apply the local optimization procedure to each of them until the solution stops changing. Let $x'_1, \ldots, x'_n$ denote the resulting solutions. From these solutions, we select the nondominated ones. Let R be the Pareto curve approximation constructed from the latter solutions.

Next, we apply Algorithm 1 to the pairs of locally optimized solutions to solve the following problems:

$$Z_2(W(x'_1, x'_2)), \ldots, Z_2(W(x'_{n-1}, x'_n)).$$

From all the solutions obtained, we choose the nondominated ones. Let Q be the Pareto curve approximation constructed from the latter solutions.

We compare the number of points not dominated by the corresponding approximation and differing by the criterion value from all points of this approximation. We introduce the number of points not dominated by the approximation A as follows: $B(A) = \big|\{x | x$ is a feasible solution of the problem Z_2, $\nexists x' \in A$ such that $C(x') \leqslant C(x)$ and $D(x') \leqslant D(x)\}\big|$. For each test, $B(R)$ and $B(Q)$ were computed by checking the entire set of feasible solutions for the nonexistence of a dominant or equal-criteria-value solution in the corresponding approximation (via complete enumeration). The tests were carried out on dimensions $n = 6, 7, 8$, with 50 tests for each dimension.

The results of the first computational experiment are given in Table 1. Clearly, the application of Algorithm 1 reduced the number of nondominated points by 6.57% on average.

Table 1

Problem dimension	$\frac{B(R)-B(Q)}{B(R)}100\%$
6	2.14%
7	8.03%
8	9.54%

Experiment 2. Construct the matrices c_{ijk} and d_{ijk} in the following way: for each index $i \in I \cup J \cup K$, generate a random point p on the XY plane so that p_x and p_y are integer and uniformly distributed on the closed interval $[0, 2^{32} - 1]$. Then $c_{ijk} = dist(i, j) + dist(j, k) + dist(i, k)$, $d_{ijk} = -\max(dist(i, j), dist(j, k), dist(i, k))$, where $dist(a, b)$ indicates the Manhattan distance between points a and b. Determine d_{ijk} by analogy.

We construct n^3 random feasible solutions $x_1, \ldots, x_{n^3}$ of the problem Z_2, and apply one iteration of the local optimization procedure to each of them. Let $x'_1, x'_2, \ldots, x'_{n^3}$ denote the resulting solutions. From these solutions, we select the nondominated ones. Let R be the Pareto curve approximation constructed from the latter solutions.

Next, we apply Algorithm 1 to the pairs of locally optimized solutions to solve the following problems:

$$Z_2(W(x'_1, x'_2)), \ldots, Z_2(W(x'_{n^3-1}, x'_{n^3})).$$

From all the solutions obtained, we choose the nondominated ones. Let Q be the Pareto curve approximation constructed from the latter solutions.

We compare the number of points not dominated by the corresponding approximation and differing by the criterion value from all points of this approximation.

The tests were carried out on dimensions $n = 6, 7, 8$, with 50 tests for each dimension.

Table 2

Problem dimension	$\frac{B(R)-B(Q)}{B(R)}100\%$
6	11.44%
7	18.08%
8	11.71%

The results of the second computational experiment are given in Table 2. Clearly, the application of Algorithm 1 reduced the number of nondominated points by 13.74% on average.

5. CONCLUSIONS

This paper has proposed an algorithm for constructing Pareto optimal solutions in the problem of combining two feasible solutions of the two-criteria three-index assignment problem. The correctness of the algorithm has been proved, and a quadratic estimate of its complexity has been obtained. This algorithm can be an additional step in heuristic approaches for estimating the Pareto domain of the two-criteria three-index assignment problem. According to the results of computational experiments provided, post-processing by the proposed optimal combination algorithm improves the quality of approximate solutions.

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